南京工程学院

第一章 Fourier 变换

转换

- 1° 对数变换(代数运算)
- 2° 坐标变换(解析几何)
- 3° 保角变换(复变函数)
- 4° 积分变换
  - $F(\alpha)=\int_a^b f(t)K(t,\ \alpha)\mathrm{d}t\Big($函数类A：$f(t)\rightarrow$函数类B：$F(\alpha)\Big)$
  - $K(t,\ \alpha)$：积分变换的核，二元函数
  - $F(\omega)=\int_{-\infty}^{+\infty} f(t)e^{-j\omega t}\mathrm{d}t\Big(K(t,\ \alpha)=e^{-j\omega t}\Big)$
  - $F(s)=\int_0^{+\infty} f(t)e^{-st}\mathrm{d}t\Big(K(t,\ \alpha)=e^{-st}\Big)$

Fourier 级数

三角形式

$$f_T(t)=\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos n\omega t+b_n\sin n\omega t)$$

$$1.\omega=\frac{2\pi}{T}$$

$$2.a_0=\frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}}f_T(t)dt$$

$$3.a_n=\frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}}f_T(t)\cos n\omega t dt(n=1,2,3,\cdots)$$

$$4.b_n=\frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}}f_T(t)\sin n\omega t dt(n=1,2,3,\cdots)$$

复指数形式

$$f_T(t)=\frac{1}{T}\sum_{n=-\infty}^{+\infty}\left[\int_{-\frac{T}{2}}^{\frac{T}{2}}f_T(\tau)e^{-j\omega_n\tau}d\tau\right]e^{j\omega_n t}$$

1.*Euler*公式：$e^{j\theta}=\cos\theta+j\sin\theta$

$$\cos\theta=\frac{e^{j\theta}+e^{-j\theta}}{2}$$

$$\sin\theta=-j\frac{e^{j\theta}-e^{-j\theta}}{2}$$

2.

$$f_T(t)=\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos n\omega t+b_n\sin n\omega t)$$

$$=\frac{a_0}{2}+\sum_{n=1}^{\infty}[a_n\frac{e^{jn\omega t}+e^{-jn\omega t}}{2}+b_n\frac{e^{jn\omega t}-e^{-jn\omega t}}{2j}]$$

$$=\frac{a_0}{2}+\sum_{n=1}^{\infty}[\frac{a_n-jb_n}{2}e^{jn\omega t}+\frac{a_n+jb_n}{2}e^{-jn\omega t}]$$

$\Leftrightarrow$

$$1.c_0=\frac{a_0}{2}=\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}}f_T(t)dt;\omega_n=n\omega(n=0,\pm1,\pm2,\cdots)$$

$$2.c_n=\frac{a_n-jb_n}{2}=\frac{1}{T}\left[\int_{-\frac{T}{2}}^{\frac{T}{2}}f_T(t)\cos n\omega t dt-j\int_{-\frac{T}{2}}^{\frac{T}{2}}f_T(t)\sin n\omega t dt\right]$$

$$=\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}}f_T(t)\left[\cos n\omega t-j\sin n\omega t\right]dt$$

$$=\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}}f_T(t)e^{-jn\omega t}dt(n=1,2,3,\cdots)$$

$$3.c_{-n}=\frac{a_n+jb_n}{2}=\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}}f_T(t)e^{jn\omega t}dt(n=1,2,3,\cdots)$$

$$\Rightarrow c_n=\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}}f_T(t)e^{-jn\omega t}\mathrm{d}t(n=0,\pm1,\pm2,\cdots)$$

$$4.f_T(t)=c_0+\sum_{n=1}^{\infty}\left[c_ne^{j\omega_n t}+c_{-n}e^{-j\omega_n t}\right]=\sum_{n=-\infty}^{+\infty}c_ne^{j\omega_n t}$$

$$
\begin{cases}
1.\ \lim\limits_{T\to+\infty} f_T(t)=f(t)=\lim\limits_{T\to+\infty}\dfrac{1}{T}\sum\limits_{n=-\infty}^{+\infty}\left[\int_{-\frac{T}{2}}^{\frac{T}{2}} f_T(\tau)e^{-j\omega_n\tau}d\tau\right]e^{j\omega_n t}\\
\quad=\lim\limits_{\Delta\omega_n\to0}\dfrac{1}{2\pi}\sum\limits_{n=-\infty}^{+\infty}\left[\int_{-\frac{T}{2}}^{\frac{T}{2}} f_T(\tau)e^{-j\omega_n\tau}d\tau\right]e^{j\omega_n t}\Delta\omega_n\left(\Delta\omega_n=\omega_n-\omega_{n-1}=\dfrac{2\pi}{T}\right)\\
\quad=\lim\limits_{\Delta\omega_n\to0}\sum\limits_{n=-\infty}^{+\infty}\Phi_T(\omega_n)\Delta\omega_n\left(\Phi_T(\omega_n)=\dfrac{1}{2\pi}\sum\limits_{n=-\infty}^{+\infty}\left[\int_{-\frac{T}{2}}^{\frac{T}{2}} f_T(\tau)e^{-j\omega_n\tau}d\tau\right]e^{j\omega_n t}\right)\\
\quad=\int_{-\infty}^{+\infty}\Phi(\omega_n)d\omega_n\left(\Phi(\omega_n)=\dfrac{1}{2\pi}\left[\int_{-\infty}^{+\infty} f(\tau)e^{-j\omega_n\tau}d\tau\right]e^{j\omega_n t}(\Delta\omega_n\to0\Leftrightarrow T\to+\infty,\Phi_T(\omega_n)\to\Phi(\omega_n))\right)\\
\quad=\int_{-\infty}^{+\infty}\Phi(\omega)d\omega\\
\quad=\dfrac{1}{2\pi}\int_{-\infty}^{+\infty}\left[\int_{-\infty}^{+\infty} f(\tau)e^{-j\omega\tau}d\tau\right]e^{j\omega t}d\omega\\
\text{Fourier积分公式：}f(t)=\dfrac{1}{2\pi}\int_{-\infty}^{+\infty}\left[\int_{-\infty}^{+\infty} f(\tau)e^{-j\omega\tau}d\tau\right]e^{j\omega t}d\omega\\
\text{间断点t处：}\dfrac{f(t+0)+f(t-0)}{2}
\end{cases}
$$

$$
\text{Fourier积分定理}
\begin{cases}
\text{条件}\begin{cases}
1.f(t)\text{在任一有限区间上满足}\\
\quad\text{Dirichlet条件}\left\{\text{函数在}\left[-\dfrac{T}{2},\dfrac{T}{2}\right]\text{上满足}\begin{cases}\text{1.连续或只有有限个第一类间断点}\\ \text{2.只有有限个极值点}\end{cases}\right.\\
2.f(t)\text{在}(-\infty,+\infty)\text{上绝对可积}\left(\int_{-\infty}^{+\infty}|f(t)|dt\text{收敛}\right)
\end{cases}\\
\text{表达式：}f(t)=\dfrac{1}{2\pi}\int_{-\infty}^{+\infty}\left[\int_{-\infty}^{+\infty} f(\tau)e^{-j\omega\tau}d\tau\right]e^{j\omega t}d\omega\text{（复数形式）}\\
\text{间断点}t\text{处：}\dfrac{f(t+0)+f(t-0)}{2}
\end{cases}
$$

三角形式

$$f(t)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\left[\int_{-\infty}^{+\infty}f(\tau)e^{-j\omega\tau}d\tau\right]e^{j\omega t}d\omega$$

$$=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\left[\int_{-\infty}^{+\infty}f(\tau)e^{j\omega(t-\tau)}d\tau\right]d\omega$$

$$=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\left[\int_{-\infty}^{+\infty}f(\tau)\cos\omega(t-\tau)d\tau+j\int_{-\infty}^{+\infty}f(\tau)\sin\omega(t-\tau)d\tau\right]d\omega$$

$$=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\left[\int_{-\infty}^{+\infty}f(\tau)\cos\omega(t-\tau)d\tau\right]d\omega\left(\int_{-\infty}^{+\infty}\left[\int_{-\infty}^{+\infty}f(\tau)\sin\omega(t-\tau)d\tau\right]d\omega=0(\text{奇函数})\right)$$

$$f(t)=\frac{1}{\pi}\int_{0}^{+\infty}\left[\int_{-\infty}^{+\infty}f(\tau)\cos\omega(t-\tau)d\tau\right]d\omega(\text{偶函数})$$

$$f(t)=\frac{1}{\pi}\int_{0}^{+\infty}\left[\int_{-\infty}^{+\infty}f(\tau)(\cos\omega t\cos\omega\tau+\sin\omega t\sin\omega\tau)d\tau\right]d\omega$$

$\Rightarrow$

正弦积分：$f(t)=\frac{2}{\pi}\int_{0}^{+\infty}\left[\int_{0}^{+\infty}f(\tau)\sin\omega\tau d\tau\right]\sin\omega t d\omega(f(t)=\text{奇函数})$

余弦积分：$f(t)=\frac{2}{\pi}\int_{0}^{+\infty}\left[\int_{0}^{+\infty}f(\tau)\cos\omega\tau d\tau\right]\cos\omega t d\omega(f(t)=\text{偶函数})$

奇展拓$\Rightarrow$正弦积分展开式；偶展拓$\Rightarrow$余弦积分展开式

Fourier变换

$f(t)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\left[\int_{-\infty}^{+\infty}f(\tau)e^{-j\omega\tau}d\tau\right]e^{j\omega t}d\omega$[积分变换, 逆变换]

$F(\omega)=\int_{-\infty}^{+\infty}f(t)e^{-j\omega t}dt$ （Fourier变换）（象函数ω）$F(\omega)=\mathbb{F}[f(t)]$

$f(t)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}F(\omega)e^{j\omega t}d\omega$(逆/积分变换)（象原函数$t$）$f(t)=\mathbb{F}^{-1}[F(\omega)]$

奇偶性相同

$f(t)=$奇：

$F_s(\omega)=\int_{0}^{+\infty}f(t)\sin\omega t dt$ （正弦变换）$F_s(\omega)=\mathbb{F}_S[f(t)]$

$f(t)=\frac{2}{\pi}\int_{0}^{+\infty}F_s(\omega)\sin\omega t d\omega$（正弦逆变换）$f(t)=\mathbb{F}_S^{-1}[F_s(\omega)]$

$f(t)=$偶：

$F_c(\omega)=\int_{0}^{+\infty}f(t)\cos\omega t dt$ （余弦变换）$F_c(\omega)=\mathbb{F}_C[f(t)]$

$f(t)=\frac{2}{\pi}\int_{0}^{+\infty}F_c(\omega)\cos\omega t d\omega$（余弦逆变换）$f(t)=\mathbb{F}_C^{-1}[F_c(\omega)]$

指数衰减函数

1. $f(t)=\begin{cases}0, & t<0\\ e^{-\beta t}, & t\geq 0\end{cases},\quad \beta>0$

2. Fourier变换

$$F(\omega)=\mathbb{F}\left[f(t)\right]=\int_{-\infty}^{+\infty}f(t)e^{-j\omega t}dt=\int_{0}^{+\infty}e^{-\beta t}e^{-j\omega t}dt=\int_{0}^{+\infty}e^{-(\beta+j\omega)t}dt=\frac{1}{\beta+j\omega}=\frac{\beta-j\omega}{\beta^2+\omega^2}$$

Fourier积分变换

$$f(t)=\mathbb{F}^{-1}\left[F(\omega)\right]=\frac{1}{2\pi}\int_{-\infty}^{+\infty}F(\omega)e^{j\omega t}d\omega=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{\beta-j\omega}{\beta^2+\omega^2}e^{j\omega t}d\omega=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{\beta\cos\omega t+\omega\sin\omega t}{\beta^2+\omega^2}d\omega=\frac{1}{\pi}\int_{0}^{+\infty}\frac{\beta\cos\omega t+\omega\sin\omega t}{\beta^2+\omega^2}d\omega$$

$$\int_{0}^{+\infty}\frac{\beta\cos\omega t+\omega\sin\omega t}{\beta^2+\omega^2}d\omega=\begin{cases}0, & t<0\\ \frac{\pi}{2}, & t=0\\ \pi e^{-\beta t}, & t>0\end{cases}$$

$\delta-$函数

$$\int_{-\infty}^{+\infty}\delta(t)f(t)dt=\lim_{\varepsilon\to 0}\int_{-\infty}^{+\infty}\delta_{\varepsilon}(t)f(t)dt$$

$$\delta_{\varepsilon}(t)=\begin{cases}0, & t<0;\\ \frac{1}{\varepsilon}, & 0\leq t\leq\varepsilon;\\ 1, & t\geq\varepsilon.\end{cases}\Rightarrow\delta_{\varepsilon}(t)\ \underset{\varepsilon\to 0}{\overset{弱}{\Rightarrow}}\ \delta(t);(\lim_{\varepsilon\to 0}\delta_{\varepsilon}(t)=\delta(t))\begin{cases}\int_{-\infty}^{+\infty}\delta_{\varepsilon}(t)dt=\int_{0}^{\varepsilon}\frac{1}{\varepsilon}dt=1(\varepsilon>0)\\ \int_{-\infty}^{+\infty}\delta(t)dt=1\end{cases}$$

筛选性质

$$\int_{-\infty}^{+\infty}\delta(t)f(t)dt=f(0)\Rightarrow\int_{-\infty}^{+\infty}\delta(t-t_0)f(t)dt=f(t_0)$$

$$\int_{-\infty}^{+\infty}\delta(t)f(t)dt=\lim_{\varepsilon\to 0}\int_{-\infty}^{+\infty}\delta_{\varepsilon}(t)f(t)dt=\lim_{\varepsilon\to 0}\int_{0}^{+\infty}\frac{1}{\varepsilon}f(t)dt=\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_{0}^{\varepsilon}f(t)dt=\lim_{\varepsilon\to 0}f(\theta\varepsilon)(0<\theta<1)$$

其它性质

1. $\delta-$函数是偶函数，即$\delta(t)=\delta(-t)$;

2. $\int_{-\infty}^{t}\delta(\tau)d\tau=u(t),\ \frac{d}{dt}u(t)=\delta(t)$, 其中$u(t)=\begin{cases}0, & t<0,\\ 1, & t>0\end{cases}$ 单位阶跃函数

3. 若$f(t)$为无穷次可微函数，则$\int_{-\infty}^{+\infty}\delta'(t)f(t)dt=-f'(0)$

$$\left(\int_{-\infty}^{+\infty}\delta^{(n)}(t)f(t)dt=(-1)^n f^{(n)}(0)\right)$$

$\delta-$函数Fourier变换:

$$F(\omega)=\mathbb{F}\left[\delta(t)\right]=\int_{-\infty}^{+\infty}\delta(t)e^{-j\omega t}dt=e^{-j\omega t}\big|_{t=0}=1\ \{\delta(t)\Leftrightarrow 1$$

$$F(\omega)=\mathbb{F}\left[\delta(t-t_0)\right]=\int_{-\infty}^{+\infty}\delta(t-t_0)e^{-j\omega t}dt=e^{-j\omega t}\big|_{t=t_0}=e^{-j\omega t_0}\ \{\delta(t-t_0)\Leftrightarrow e^{-j\omega t_0}$$

单位阶跃函数

1. $u(t)=\begin{cases}0, t<0\\ 1, t>0\end{cases}$

2. Fourier变换：$F(\omega)=\dfrac{1}{j\omega}+\pi\delta(\omega)$

3. Fourier逆（积分）变换

$$\begin{aligned} f(t)&=\mathbb{F}^{-1}\left[F(\omega)\right]\\ &=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\left[\frac{1}{j\omega}+\pi\delta(\omega)\right]e^{j\omega t}d\omega\\ &=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\pi\delta(\omega)e^{j\omega t}d\omega+\frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{e^{j\omega t}}{j\omega}d\omega\\ &=\frac{1}{2}\int_{-\infty}^{+\infty}\delta(\omega)e^{j\omega t}d\omega+\frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{\sin\omega t}{\omega}d\omega\\ &=\frac{1}{2}+\frac{1}{\pi}\int_{0}^{+\infty}\frac{\sin\omega t}{\omega}d\omega=\begin{cases}\dfrac{1}{2}+\dfrac{1}{\pi}\left(-\dfrac{\pi}{2}\right)=0, t<0\\ \dfrac{1}{2}+\dfrac{1}{\pi}\bullet\dfrac{\pi}{2}=1, \quad t>0\end{cases}\end{aligned}$$

Dirichlet积分 $\int_0^{+\infty}\frac{\sin\omega}{\omega}d\omega=\frac{\pi}{2}\Rightarrow\int_0^{+\infty}\frac{\sin\omega t}{\omega}d\omega=\begin{cases}-\dfrac{\pi}{2}, & t<0\\ 0, & t=0\\ \dfrac{\pi}{2}, & t>0\end{cases}$

当t<0时，令 $u=-t\omega\Rightarrow\int_0^{+\infty}\frac{\sin\omega t}{\omega}d\omega=\int_0^{+\infty}\frac{\sin(-u)}{u}du=-\int_0^{+\infty}\frac{\sin u}{u}du=-\frac{\pi}{2}$

$$u(t)=\frac{1}{2}+\frac{1}{\pi}\int_0^{+\infty}\frac{\sin\omega t}{\omega}d\omega\,(t\neq 0)\Rightarrow\begin{cases}\begin{cases}F(\omega)=2\pi\delta(\omega)\Leftrightarrow f(t)=\dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{+\infty}2\pi\delta(\omega)e^{j\omega t}d\omega=1\\ 2\pi\delta(\omega-\omega_0)\Leftrightarrow e^{j\omega_0 t}\end{cases}\\ \begin{cases}\displaystyle\int_{-\infty}^{+\infty}e^{-j\omega t}d\omega=2\pi\delta(\omega)\\ \displaystyle\int_{-\infty}^{+\infty}e^{-j(\omega-\omega_0)t}d\omega=2\pi\delta(\omega-\omega_0)\end{cases}\end{cases}$$

化三角为 $e^{-j\omega t}$ 求解Fourier变换

非周期函数频谱

第n次谐波

三角形式：$a_n\cos\omega_n t+b_n\sin\omega_n t=A_n\sin(\omega_n t+\varphi_n)=\sqrt{a_n^2+b_n^2}\sin(\omega_n t+\varphi_n)$

复数形式：$c_n e^{j\omega_n t}+c_{-n}e^{-j\omega_n t}$ $\begin{cases}c_n=\dfrac{a_n-jb_n}{2}, \quad c_{-n}=\dfrac{a_n+jb_n}{2};\\ |c_n|=|c_{-n}|=\dfrac{1}{2}\sqrt{a_n^2+b_n^2}\,(n=0,1,2,\cdots)\\ \mathrm{A_n}=2\sqrt{a_n^2+b_n^2}\end{cases}$

$f(t)=\dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{+\infty}\mathrm{F}(\omega)e^{j\omega_n t}d\omega$，其中 $\mathrm{F}(\omega)=\displaystyle\int_{-\infty}^{+\infty}f(t)e^{-j\omega_n t}dt$

*Fourier*变换(原函数)性质

1.线性性质

设$F_1(\omega)=\mathbb{F}[f_1(t)]$，$F_2(\omega)=\mathbb{F}[f_2(t)]$，

则$\mathbb{F}[\alpha f_1(t)\pm\beta f_2(t)]=\alpha\mathbb{F}[f_1(t)]\pm\beta\mathbb{F}[f_2(t)]=\alpha F_1(\omega)\pm\beta F_2(\omega)$

$\mathbb{F}^{-1}[\alpha F_1(\omega)\pm\beta F_2(\omega)]=\alpha\mathbb{F}^{-1}[F_1(\omega)]\pm\beta\mathbb{F}^{-1}[F_2(\omega)]=\alpha f_1(t)\pm\beta f_2(t)$

2.位移性质

$\mathbb{F}[f(t\pm t_0)]=e^{\pm j\omega t_0}\mathbb{F}[f(t)]$

$\mathbb{F}^{-1}[F(\omega\mp\omega_0)]=f(t)e^{\pm jt\omega_0}$

3.微分性质

如果$f(t)$在$(-\infty,+\infty)$上连续或只有有限个可去间断点，

且当$|t|\to+\infty$时，$f(t)\to 0$,则

$\mathbb{F}[f'(t)]=j\omega\mathbb{F}[f(t)]=j\omega F(\omega)$;

$F'(\omega)=\dfrac{d}{d\omega}F(\omega)=\mathbb{F}[-jtf(t)]\Rightarrow\mathbb{F}[tf(t)]=jF'(\omega)$

推理

如果$f^{(k)}(t)$在$(-\infty,+\infty)$上连续或只有有限个可去间断点，

且$\lim\limits_{|t|\to+\infty}f^{(k)}(t)=0$, k=0, 1, 2,⋯, n−1时，则

$\mathbb{F}[f^{(n)}(t)]=(j\omega)^n\mathbb{F}[f(t)]$;

(象函数微分性质) $F^{(n)}(\omega)=\dfrac{d^n}{d\omega^n}F(\omega)=(-j)^n\mathbb{F}[t^nf(t)]$

4.积分性质

如果当$t\to+\infty$时，$g(t)=\int_{-\infty}^{t}f(t)dt\to 0$,则

$\mathbb{F}\left[\int_{-\infty}^{t}f(t)dt\right]=\dfrac{1}{j\omega}\mathbb{F}[f(t)]$

5*乘积定理

6*能量积分

卷积

- 定义：$f_1(t)*f_2(t)=\int_{-\infty}^{+\infty}f_1(\tau)f_2(t-\tau)d\tau$
- 不等性：$|f_1(t)*f_2(t)|\le|f_1(t)|*|f_2(t)|$
- 交换律：$f_1(t)*f_2(t)=f_2(t)*f_1(t)$
- 结合律：$f_1(t)*[f_2(t)*f_3(t)]=[f_1(t)*f_2(t)]*f_3(t)$
- 分配律：
$$\begin{cases} f_1(t)*\left[f_2(t)+f_3(t)\right]=f_1(t)*f_2(t)+f_1(t)*f_3(t)\\ [f_1(t)+f_2(t)]*\left[g_1(t)+g_2(t)\right]\\ =f_1(t)*g_1(t)+f_1(t)*g_2(t)+f_2(t)*g_1(t)+f_2(t)*g_2(t)\end{cases}$$
- $\frac{d}{dt}[f_1(t)*f_2(t)]=\frac{d}{dt}f_1(t)*f_2(t)=f_1(t)*\frac{d}{dt}f_2(t)=f_2(t)*\frac{d}{dt}f_1(t)$
- $$\begin{cases} f(t)*\delta(t)=f(t)\\ f(t)*\delta(t-t_0)=f(t-t_0)\\ f(t)*\delta'(t)=f'(t)\end{cases}$$
- $f(t)*u(t)=\int_{-\infty}^{t}f(\tau)d\tau$
- 卷积定理：
  - 假定$f_1(t)$，$f_2(t)$都满足*Fourier*积分定理中的条件，
  - 且$F_1(\omega)=\mathbb{F}\left[f_1(t)\right]$，$F_2(\omega)=\mathbb{F}\left[f_2(t)\right]$则
  - $\mathbb{F}\left[f_1(t)*f_2(t)\right]=F_1(\omega)\bullet F_2(\omega)$或$\mathbb{F}^{-1}\left[F_1(\omega)\bullet F_2(\omega)\right]=f_1(t)*f_2(t)$
  - $\mathbb{F}\left[f_1(t)\bullet f_2(t)\right]=\frac{1}{2\pi}F_1(\omega)*F_2(\omega)$
- 推广：
$$\begin{cases} \mathbb{F}\left[f_1(t)*f_2(t)*\cdots*f_n(t)\right]=F_1(\omega)\bullet F_2(\omega)\bullet\cdots\bullet F_n(\omega)\\ \mathbb{F}\left[f_1(t)\bullet f_2(t)\bullet\cdots\bullet f_n(t)\right]=\frac{1}{(2\pi)^{n-1}}F_1(\omega)*F_2(\omega)*\cdots*F_n(\omega)\end{cases}$$

第二章 Laplace 变换

Laplace

$$G_\beta(\omega)=\int_{-\infty}^{+\infty}\varphi(t)u(t)\mathrm{e}^{-\beta t}\mathrm{e}^{-j\omega t}dt=\int_{0}^{+\infty}f(t)\mathrm{e}^{-(\beta+j\omega)t}dt=\int_{0}^{+\infty}f(t)\mathrm{e}^{-St}dt$$

$$F(s)=G_\beta(\frac{s-\beta}{j})=\int_{0}^{+\infty}f(t)\mathrm{e}^{-st}dt\qquad(Laplace\text{变换})$$

$$F(s)=\mathbb{L}\left[f(t)\right]\leftrightarrow f(t)=\mathbb{L}^{-1}\left[F(s)\right]$$

$Laplace$变换存在条件
1. 在$t\geq 0$的任一有限区间上分段连续；
2. 当$t\to+\infty$时，$f(t)$的增长速度不超过某一指数函数，亦即存在常数$M>0$及$\mathrm{c}\geq 0$, 使得 $\left|f(t)\right|\leq Me^{ct}, 0\leq t\leq+\infty$

$\Rightarrow F$

周期函数$Laplace$变换公式：$F(s)=\mathbb{L}\left[f(t)\right]=\dfrac{1}{1-e^{-sT}}\displaystyle\int_0^T f(t)\mathrm{e}^{-st}dt\qquad(\mathrm{Re}(s)>0)$

$$\begin{cases}\mathbb{L}_+\left[f(t)\right]=\int_{0^+}^{+\infty}f(t)\mathrm{e}^{-st}dt\\ \mathbb{L}_-\left[f(t)\right]=\int_{0^-}^{+\infty}f(t)\mathrm{e}^{-st}dt=\int_{0^-}^{0^+}f(t)\mathrm{e}^{-st}dt+\mathbb{L}_+\left[f(t)\right]\end{cases}\Rightarrow\begin{cases}1.f(t)\text{在t=0附近有界}:\int_{0^-}^{0^+}f(t)\mathrm{e}^{-st}dt=0,\ \mathbb{L}_-\left[f(t)\right]=\mathbb{L}_+\left[f(t)\right]\\ 2.f(t)\text{在t=0处包含了脉冲函数}:\int_{0^-}^{0^+}f(t)\mathrm{e}^{-st}dt\neq 0,\ \mathbb{L}_-\left[f(t)\right]\neq\end{cases}$$

$\mathbb{L}aplace$性质

1.线性

$$\mathbb{L}[\alpha f_1(t)+\beta f_2(t)]=\alpha\mathbb{L}[f_1(t)]+\beta\mathbb{L}[f_2(t)]$$

$$\mathbb{L}^{-1}[\alpha F_1(s)+\beta F_2(s)]=\alpha\mathbb{L}^{-1}[F_1(s)]+\beta\mathbb{L}^{-1}[F_2(s)]$$

2.微分 $\mathbb{L}\left[f'(t)\right]=sF(s)-f(0)$

推论

$$\mathbb{L}\left[f^{(n)}(t)\right]=s^nF(s)-s^{n-1}f(0)-s^{n-2}f'(0)-\cdots f^{(n-1)}(0)$$

$$=s^nF(s)-\sum_{i=0}^{n-1}s^{n-1-i}f^{(i)}(0)\qquad \left(Re(s)>\mathrm{c}\right)$$

特别的：$f(0)=f'(0)=\cdots=f^{(n-1)}(0)=0\Rightarrow\mathbb{L}\left[f^{(n)}(t)\right]=s^nF(s)$

象函数微分性质 $\begin{cases}F'(s)=-\mathbb{L}[tf(t)]\\F^{(n)}(s)=(-1)^n\mathbb{L}\left[t^nf(t)\right]\end{cases}\Leftrightarrow\mathbb{L}\left[t^nf(t)\right]=(-1)^nF^{(n)}(s)$

$$F^{(n)}(s)=(-1)^n\mathbb{L}\left[t^nf(t)\right]$$

$$=\mathbb{L}\left[(-t)^nf(t)\right]=\int_0^{+\infty}(-t)^nf(t)e^{-st}dt$$

$$=\int_0^{+\infty}\frac{d^n}{ds^n}f(t)e^{-st}dt=\frac{d^n}{ds^n}\int_0^{+\infty}f(t)e^{-st}d$$

$=F^{(n)}(s)$　　(变换积分和微分顺序

3.积分

若$\mathbb{L}[f(t)]=F(s)$，则$\mathbb{L}\left[\int_0^tf(t)dt\right]=\frac{1}{\mathrm{s}}F(s)$

推广：$\mathbb{L}\left\{\underbrace{\int_0^tdt\int_0^tdt\cdots\int_0^tf(t)dt}_{n次}\right\}=\frac{1}{s^n}F(s)$

证明：

1. 设$h(t)=\int_0^tf(t)dt$，则$h'(t)=f(t),h(0)=0$

   $\mathbb{L}\left[h'(t)\right]=s\mathbb{L}\left[h(t)\right]-h(0)=s\mathbb{L}\left[h(t)\right]$

   即：$\mathbb{L}\left[h(t)\right]=\frac{1}{s}\mathbb{L}\left[h'(t)\right]$

2. 令$f_1=1$,则

   $f(t)*f_1(t)=\int_0^tf(\tau)\bullet1d\tau=\int_0^tf(t)dt$

   $\mathbb{L}\left[\int_0^tf(t)dt\right]=\mathbb{L}\left[f(t)*f_1(t)\right]=\mathbb{L}\left[f(t)\right]\bullet\mathbb{L}\left[f_1(t)\right]$

   $=\frac{1}{s}\mathbb{L}\left[f(t)\right]$

若$\mathbb{L}[f(t)]=F(s)$，则$\mathbb{L}\left[\frac{f(t)}{t}\right]=\int_s^\infty F(s)ds$或$f($

推广：$\mathbb{L}\left[\frac{f(t)}{t^n}\right]=\underbrace{\int_s^\infty ds\int_s^\infty ds\cdots\int_s^\infty F(s)ds}_{n次}$

4.位移 若$\mathbb{L}[f(t)]=F(s)$，则$\mathbb{L}\left[e^{at}f(t)\right]=F(s-a)$　　$\left(\mathrm{Re}(s-a)>c\right)$

5.延迟

若$\mathbb{L}[f(t)]=F(s)$，又$t<0$时$f(t)=0$，则对任一非负实数τ，有

则　或 $\begin{cases}\mathbb{L}\left[f(t-\tau)\right]=e^{-s\tau}F(s)\\\mathbb{L}^{-1}\left[e^{-s\tau}F(s)\right]=f(t-\tau)u(t-\tau)\end{cases}\Leftrightarrow\mathbb{L}\left[f(t-\tau)u(t-\tau)\right]=e^{-s\tau}F(s)$

6.相似 $\mathbb{L}[f(at)]=\frac{1}{a}F(\frac{s}{a})$

Laplace逆变换

1. $f(t)u(t)e^{-\beta t}=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\left[\int_{-\infty}^{+\infty}f(\tau)u(\tau)e^{-\beta\tau}e^{-j\omega\tau}d\tau\right]e^{j\omega t}d\omega=\frac{1}{2\pi}\int_{-\infty}^{+\infty}e^{j\omega t}d\omega\left[\int_{0}^{+\infty}f(\tau)e^{-(\beta+j\omega)\tau}d\tau\right]=\frac{1}{2\pi}\int_{-\infty}^{+\infty}F(\beta+$

2. $f(t)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}F(\beta+j\omega)e^{(\beta+j\omega)t}d\omega=\frac{1}{2\pi j}\int_{\beta-j\infty}^{\beta+j\infty}F(s)e^{st}ds,t>0\left(\Leftrightarrow F(s)=\int_{0}^{+\infty}f(t)e^{-st}dt\right)$

3. 定理 若$s_1,s_2,\cdots,s_n$是函数$F(s)$的所有奇点（适当选取β使这些奇点全在$\mathrm{Re}(s)<\beta$的范围内），且$\lim\limits_{s\to\infty}F(s)=$

则有$f(t)=\frac{1}{2\pi j}\int_{\beta-j\infty}^{\beta+j\infty}F(s)e^{st}ds=\sum_{k=1}^{n}\operatorname*{Res}_{s=s_k}\left[F(s)e^{st}\right],t>0$

卷积

定义：$f_1(t)*f_2(t)=\int_{-\infty}^{+\infty}f_1(\tau)f_2(t-\tau)=\int_{-\infty}^{0}f_1(\tau)f_2(t-\tau)+\int_{0}^{t}f_1(\tau)f_2(t-\tau)+\int_{t}^{+\infty}f_1(\tau)f_2(t-\tau)$

$=\int_0^t f_1(\tau)f_2(t-\tau)dt$

卷积定理：

假定$f_1(t)$，$f_2(t)$都满足$\mathbb{L}$aplase变换存在定理中的条件，且$F_1(\omega)=\mathbb{L}[f_1(t)]$，$F_2(\omega)=\mathbb{L}[f_2(t)]$则

$\mathbb{L}[f_1(t)*f_2(t)]=F_1(s)\bullet F_2(s)=\mathbb{L}[f_1(t)]\bullet\mathbb{L}[f_2(t)]$

或 $\mathbb{L}^{-1}[F_1(s)\bullet F_2(s)]=\mathbb{L}^{-1}[F_1(s)]\bullet\mathbb{L}^{-1}[F_2(s)]=f_1(t)*f_2(t)$

推广：$\{\mathbb{F}[f_1(t)*f_2(t)*\cdots*f_n(t)]=F_1(\omega)\bullet F_2(\omega)\bullet\cdots\bullet F_n(\omega)=\mathbb{L}[f_1(t)]\bullet\mathbb{L}[f_2(t)]\bullet\cdots\bullet\mathbb{L}[f_n(t)]$

$$
Fourier\begin{cases}
1.\underset{\text{指数衰减函数}}{\underline{f(t)=\begin{cases}0, & t<0\\ e^{-\beta t}, & t\ge 0(\beta>0)\end{cases}}} \Leftrightarrow F(\omega)=\mathbb{F}\left[f(t)\right]=\dfrac{1}{\beta+j\omega}\\
2.\begin{cases}
\underset{\text{单位脉冲函数}}{\underline{f(t)=\delta(t)}} \Leftrightarrow F(\omega)=\mathbb{F}\left[f(t)\right]=1\\
f(t)=\delta(t-c)\Leftrightarrow F(\omega)=\mathbb{F}\left[f(t)\right]=e^{-j\omega c}\\
f(t)=\delta^{'}(t)\Leftrightarrow F(\omega)=\mathbb{F}\left[f(t)\right]=j\omega\\
f(t)=\delta^{(n)}(t)\Leftrightarrow F(\omega)=\mathbb{F}\left[f(t)\right]=(j\omega)^{n}\\
f(t)=\delta^{(n)}(t-c)\Leftrightarrow F(\omega)=\mathbb{F}\left[f(t)\right]=(j\omega)^{n}e^{-j\omega c}
\end{cases}\\
3.\begin{cases}
f(t)=1\Leftrightarrow F(\omega)=\mathbb{F}\left[f(t)\right]=2\pi\delta(\omega)\\
f(t)=t\Leftrightarrow F(\omega)=\mathbb{F}\left[f(t)\right]=2\pi j\delta^{'}(\omega)\\
f(t)=t^{n}\Leftrightarrow F(\omega)=\mathbb{F}\left[f(t)\right]=2\pi j^{n}\delta^{(n)}(\omega)\\
f(t)=e^{j\alpha t}\Leftrightarrow F(\omega)=\mathbb{F}\left[f(t)\right]=2\pi\delta(\omega-\alpha)\\
f(t)=t^{n}e^{j\alpha t}\Leftrightarrow F(\omega)=\mathbb{F}\left[f(t)\right]=2\pi j^{n}\delta^{(n)}(\omega-\alpha)
\end{cases}\\
4.\begin{cases}
\underset{\text{单位函数}}{\underline{f(t)=u(t)}} \Leftrightarrow F(\omega)=\mathbb{F}\left[f(t)\right]=\dfrac{1}{j\omega}+\pi\delta(\omega)\\
f(t)=u(t-c)\Leftrightarrow F(\omega)=\mathbb{F}\left[f(t)\right]=\dfrac{1}{j\omega}e^{-j\omega c}+\pi\delta(\omega)\\
f(t)=u(t)\bullet t^{n}\Leftrightarrow F(\omega)=\mathbb{F}\left[f(t)\right]=\dfrac{n!}{(j\omega)^{n+1}}+\pi j^{n}\delta^{(n)}(\omega)\\
f(t)=u(t)\sin\alpha t\Leftrightarrow F(\omega)=\mathbb{F}\left[f(t)\right]=\dfrac{\alpha}{\alpha^{2}-\omega^{2}}+\dfrac{1}{2}j\pi\left[\delta(\omega+\omega_0)-\delta(\omega-\omega_0)\right]\\
f(t)=u(t)\cos\alpha t\Leftrightarrow F(\omega)=\mathbb{F}\left[f(t)\right]=\dfrac{j\omega}{\alpha^{2}-\omega^{2}}+\dfrac{1}{2}\pi\left[\delta(\omega+\omega_0)+\delta(\omega-\omega_0)\right]\\
f(t)=u(t)e^{j\alpha t}\Leftrightarrow F(\omega)=\mathbb{F}\left[f(t)\right]=\dfrac{1}{j(\omega-\alpha)}+\pi\delta(\omega-\alpha)\\
f(t)=u(t-c)e^{j\alpha t}\Leftrightarrow F(\omega)=\mathbb{F}\left[f(t)\right]=\dfrac{1}{j(\omega-\alpha)}e^{-j(\omega-\alpha)c}+\pi\delta(\omega-\alpha)\\
f(t)=u(t)e^{j\alpha t}\bullet t^{n}\Leftrightarrow F(\omega)=\mathbb{F}\left[f(t)\right]=\dfrac{1}{[j(\omega-\alpha)]^{n+1}}+\pi j^{n}\delta^{(n)}(\omega-\alpha)
\end{cases}\\
5.\begin{cases}
f(t)=\sin\omega_0 t\Leftrightarrow F(\omega)=\mathbb{F}\left[f(t)\right]=j\pi\left[\delta(\omega+\omega_0)-\delta(\omega-\omega_0)\right]\\
f(t)=\cos\omega_0 t\Leftrightarrow F(\omega)=\mathbb{F}\left[f(t)\right]=\pi\left[\delta(\omega+\omega_0)+\delta(\omega-\omega_0)\right]
\end{cases}
\end{cases}
$$

$$
Laplace\begin{cases}
1.\, f(t)=1\big(u(t)\big) \Leftrightarrow F(s)=\dfrac{1}{s}\begin{cases}\mathbb{L}\left[u(t)\right]=\mathbb{L}\left[u(at)\right] & a>0\\ \mathbb{L}\left[u\big(\varphi(t)\big)\right]=\mathbb{L}\left[1\right] & \varphi(t)>0\end{cases}\\
2.\, f(t)=e^{at} \Leftrightarrow F(s)=\dfrac{1}{s-a}\\
3.\begin{cases}f(t)=\sin at \Leftrightarrow F(s)=\dfrac{a}{s^2+a^2}\\ f(t)=\cos at \Leftrightarrow F(s)=\dfrac{s}{s^2+a^2}\end{cases}\begin{cases}f(t)=t\sin at \Leftrightarrow F(s)=\dfrac{2as}{(s^2+a^2)^2}\\ f(t)=t\cos at \Leftrightarrow F(s)=\dfrac{s^2-a^2}{(s^2+a^2)^2}\end{cases}\\
4.\begin{cases}f(t)=t^m\,(m>-1) \Leftrightarrow F(s)=\dfrac{m!}{s^{m+1}}\\ f(t)=e^{at}t^m\,(m>-1) \Leftrightarrow F(s)=\dfrac{m!}{(s-a)^{m+1}}\end{cases}
\end{cases}
$$

1.设 $\mathbb{F}\left[f(t)\right]=F(\omega)$，试证明（**Fourier 原函数奇偶虚实性质**）

（1） $f(t)$ 为实值函数的充要条件是 $F(-\omega)=\overline{F(\omega)}$

（2） $f(t)$ 为纯虚值函数的充要条件是 $F(-\omega)=-\overline{F(\omega)}$

证明：设

$$f(t)=f_r(t)+jf_i(t)$$

$$\Rightarrow F(\omega)=\int_{-\infty}^{+\infty}f(t)e^{-j\omega t}dt$$

$$=\int_{-\infty}^{+\infty}[f_r(t)+jf_i(t)][\cos\omega t-j\sin\omega t]dt$$

$$=\int_{-\infty}^{+\infty}[f_r(t)\cos\omega t+f_i(t)\sin\omega t]dt-j\int_{-\infty}^{+\infty}[f_r(t)\sin\omega t-f_i(t)\cos\omega t]dt$$

$$=\mathrm{Re}[F(\omega)]+j\mathrm{Im}[F(\omega)]$$

其中:$\mathrm{Re}[F(\omega)]=\int_{-\infty}^{+\infty}[f_r(t)\cos\omega t+f_i(t)\sin\omega t]dt$

$$\mathrm{Im}[F(\omega)]=-\int_{-\infty}^{+\infty}[f_r(t)\sin\omega t-f_i(t)\cos\omega t]dt$$

(1)当$f(t)=f_r(t)$，$f_i(t)=0$时，$\begin{cases}\mathrm{Re}[F(\omega)]=\int_{-\infty}^{+\infty}[f_r(t)\cos\omega t]dt\\ \mathrm{Im}[F(\omega)]=-\int_{-\infty}^{+\infty}[f_r(t)\sin\omega t]dt\end{cases}$

$F(-\omega)=\mathrm{Re}[F(-\omega)]+j\mathrm{Im}[F(-\omega)]=\mathrm{Re}[F(\omega)]-j\mathrm{Im}[F(\omega)]=\overline{F(\omega)}$

反之，若已知$F(-\omega)=\overline{F(\omega)}$，则有$\mathrm{Re}[F(-\omega)]+j\mathrm{Im}[F(-\omega)]=\mathrm{Re}[F(\omega)]-j\mathrm{Im}[F(\omega)]$

表明：$F(\omega)$实部是关于ω的偶函数，虚部是关于ω的奇函数。

因此，必定有$F(\omega)=\int_{-\infty}^{+\infty}[f_r(t)\cos\omega t]dt-j\int_{-\infty}^{+\infty}[f_r(t)\sin\omega t]dt$

即：$f(t)=f_r(t)$，$f_i(t)=0$为t的实值函数，得证。

(2)当$f(t)=jf_i(t)$，$f_r(t)=0$时，$\begin{cases}\mathrm{Re}[F(\omega)]=\int_{-\infty}^{+\infty}[f_i(t)\sin\omega t]dt\\ \mathrm{Im}[F(\omega)]=\int_{-\infty}^{+\infty}[f_i(t)\cos\omega t]dt\end{cases}$

$F(-\omega)=\mathrm{Re}[F(-\omega)]+j\mathrm{Im}[F(-\omega)]=-\mathrm{Re}[F(\omega)]+j\mathrm{Im}[F(\omega)]=-\overline{F(\omega)}$

反之，若已知$F(-\omega)=-\overline{F(\omega)}$，则有$\mathrm{Re}[F(-\omega)]+j\mathrm{Im}[F(-\omega)]=-\mathrm{Re}[F(\omega)]+j\mathrm{Im}[F(\omega)]$

表明：$F(\omega)$实部是关于ω的奇函数，虚部是关于ω的偶函数。

因此，必定有$F(\omega)=\int_{-\infty}^{+\infty}[f_i(t)\sin\omega t]dt+j\int_{-\infty}^{+\infty}[f_i(t)\cos\omega t]dt$

即：$ff(t)=jf_i(t)$，$f_r(t)=0$为t的纯虚值函数，得证。

$$2.F(\omega)=\frac{\sin\omega}{\omega}$$

$$f(t)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{\sin\omega}{\omega}e^{j\omega t}d\omega$$
$$=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{\sin\omega}{\omega}(\cos\omega t+j\sin\omega t)d\omega$$
$$=\frac{1}{\pi}\int_{0}^{+\infty}\frac{\sin\omega\cos\omega t}{\omega}d\omega$$
$$=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{\sin(1+t)\omega}{\omega}d\omega+\frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{\sin(1-t)\omega}{\omega}d\omega$$

又$\frac{1}{\pi}\int_{0}^{+\infty}\frac{\sin\omega t}{\omega}d\omega=u(t)-\frac{1}{2},(t\neq 0)$

所以$f(t)=\frac{1}{2}[u(1+t)-\frac{1}{2}+u(1-t)-\frac{1}{2}]$

$$=\frac{1}{2}[u(1+t)+u(1-t)-1],(|t|\neq 1)$$

当$t=\pm 1$时，$f(t)$应以$\frac{1}{2}[f(t+0)+f(t-0)]=\frac{1}{4}$代替，

即$f(t)=\begin{cases}\frac{1}{2}[u(1+t)+u(1-t)-1],(|t|\neq 1)\\ \frac{1}{4},(|t|=1)\end{cases}$ 或$f(t)=\begin{cases}\frac{1}{2},(|t|<1)\\ \frac{1}{4},(|t|=1)\\ 0,(|t|>1)\end{cases}$

Dirichlet积 分$\int_0^{+\infty}\frac{\sin\omega}{\omega}d\omega=\frac{\pi}{2}$

3.证明：若 $\mathbb{F}\left[e^{j\varphi(t)}\right]=F(\omega)$，其中 $\varphi(t)$ 为一实函数，则$\begin{cases}\mathbb{F}\left[\cos\varphi(t)\right]=\frac{1}{2}\left[F(\omega)+\overline{F(-\omega)}\right]\\ \mathbb{F}\left[\sin\varphi(t)\right]=\frac{1}{2j}\left[F(\omega)-\overline{F(-\omega)}\right]\end{cases}$

证明：$\begin{cases}(1)\mathrm{F}(\omega)=\mathbb{F}\left[e^{j\varphi(t)}\right]=\mathbb{F}\left[\cos\varphi(t)\right]+j\mathbb{F}\left[\sin\varphi(t)\right]\\ (2)\mathrm{F}(-\omega)=\int_{-\infty}^{+\infty}e^{j\varphi(t)}\bullet e^{-j(-\omega)t}dt=\int_{-\infty}^{+\infty}e^{j\varphi(t)}\bullet e^{j\omega t}dt\\ (3)\overline{F(-\omega)}=\int_{-\infty}^{+\infty}e^{-j\varphi(t)}\bullet e^{-j\omega t}dt=\mathbb{F}\left[e^{-j\varphi(t)}\right]=\mathbb{F}\left[\cos\varphi(t)\right]-j\mathbb{F}\left[\sin\varphi(t)\right]\end{cases}$ $\begin{cases}(1)+(3):\mathbb{F}\left[\cos\varphi(t)\right]=\frac{1}{2}\left[F(\omega)+\overline{F(-\omega)}\right]\\ (1)-(3):\mathbb{F}\left[\sin\varphi(t)\right]=\frac{1}{2j}\left[F(\omega)-\overline{F(-\omega)}\right]\end{cases}$

4. 若 $\mathbb{F}\left[f(t)\right]=F(\omega)$ ，证 明 （原 函 数 对 称 性 质）

$$f(\pm\omega)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}F(\mp t)e^{-j\omega t}dt,\text{即}\mathbb{F}\left[F(\mp t)\right]=2\pi f(\pm\omega)$$

$$\mathbb{F}\left[f(t)\right]=F(\omega)\Rightarrow\begin{cases} f(t)=\dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{+\infty}F(\omega)e^{j\omega t}d\omega & (a)\\ f(-t)=\dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{+\infty}F(\omega)e^{-j\omega t}d\omega & (b)\end{cases}$$

(a)：t与ω互换$\Rightarrow f(\omega)=\dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{+\infty}F(t)e^{j\omega t}dt$

证明：（令$t=-u$） $=\dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{+\infty}F(-u)e^{-j\omega u}du$

（令$u=t$） $=\dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{+\infty}F(-t)e^{-j\omega t}dt$

$$\left.\begin{aligned}&\text{即：}\ \mathbb{F}\left[F(-t)\right]=2\pi f(\omega)\\ &(b)\text{：}\ \mathbb{F}\left[F(t)\right]=2\pi f(-\omega)\end{aligned}\right\}\Rightarrow\mathbb{F}\left[F(\mp t)\right]=2\pi f(\pm\omega)$$

4.若 $\mathbb{F}\left[f(t)\right]=F(\omega)$ a 为非零常数，证明（原函数相似性质）$\mathbb{F}\left[f(at)\right]=\dfrac{1}{|a|}F(\dfrac{\omega}{a})$

用定义证明

证明：

$$\begin{cases}1.\text{当}a>0\text{时，}\\ \mathbb{F}\left[f(at)\right]=\displaystyle\int_{-\infty}^{+\infty}f(at)e^{-j\omega t}dt\\ (\text{令}at=u)=\dfrac{1}{a}\displaystyle\int_{-\infty}^{+\infty}f(u)e^{-j\frac{\omega}{a}t}du\\ (u\text{换成}t)=\dfrac{1}{a}\displaystyle\int_{-\infty}^{+\infty}f(t)e^{-j\frac{\omega}{a}t}dt\\ \qquad=\dfrac{1}{a}F(\dfrac{\omega}{a})\end{cases}\begin{cases}2.\text{当}a>0\text{时，}\\ \mathbb{F}\left[f(at)\right]=\displaystyle\int_{-\infty}^{+\infty}f(at)e^{-j\omega t}dt\\ (\text{令}at=u)=\dfrac{1}{a}\displaystyle\int_{+\infty}^{-\infty}f(u)e^{-j\frac{\omega}{a}t}du\\ (u\text{换成}t)=-\dfrac{1}{a}\displaystyle\int_{-\infty}^{+\infty}f(t)e^{-j\frac{\omega}{a}t}dt\\ \qquad=-\dfrac{1}{a}F(\dfrac{\omega}{a})\end{cases}\Rightarrow\mathbb{F}\left[f(at)\right]=\dfrac{1}{|a|}F(\dfrac{\omega}{a})$$

5.若 $\mathbb{F}\left[f(t)\right]=F(\omega)$，证明（象函数位移性质）

$\mathbb{F}^{-1}\left[F(\omega\mp\omega_0)\right]=e^{\pm j\omega_0 t}f(t)$

即$F(\omega\mp\omega_0)=\mathbb{F}^{-1}\left[e^{\pm j\omega_0 t}f(t)\right]$

证明：定义证

$$\begin{cases}\mathbb{F}\left[e^{\pm j\omega_0 t}f(t)\right]=\displaystyle\int_{-\infty}^{+\infty}e^{\pm j\omega_0 t}f(t)e^{-j\omega t}dt\\ \qquad=\displaystyle\int_{-\infty}^{+\infty}f(t)e^{-j(\omega\mp\omega_0)t}dt=F(\omega\mp\omega_0)\end{cases}$$

6.若 $\mathbb{F}\left[f(t)\right]=F(\omega)$，证明（象函数微分性质）$(F'(\omega))\dfrac{d}{d\omega}F(\omega)=\mathbb{F}\left[-jtf(t)\right]$

证明：

$$
\text{定义证}\begin{cases}
(F'(\omega))\dfrac{d}{d\omega}F(\omega) \\
=\dfrac{d}{d\omega}\displaystyle\int_{-\infty}^{+\infty} f(t)e^{-j\omega t}dt \\
=\displaystyle\int_{-\infty}^{+\infty}\dfrac{d}{d\omega}\left(f(t)e^{-j\omega t}\right)dt \\
=\displaystyle\int_{-\infty}^{+\infty}-jtf(t)e^{-j\omega t}dt \\
=\mathbb{F}\left[-jtf(t)\right]
\end{cases}
$$

7.若 $\mathbb{F}\left[f(t)\right]=F(\omega)$，证明（**翻转性质**） $F(-\omega)=\mathbb{F}\left[f(-t)\right]$

证明：

$$
\text{定义证}\begin{cases}
\mathbb{F}\left[f(-t)\right]=\displaystyle\int_{-\infty}^{+\infty} f(-t)e^{-j\omega t}dt \\
(-t=u)=-\displaystyle\int_{+\infty}^{-\infty} f(u)e^{-j\omega(-u)}du \\
(\text{上下限})=\displaystyle\int_{-\infty}^{+\infty} f(u)e^{-j(-\omega)u}du \\
(\text{令}u=t)=\displaystyle\int_{-\infty}^{+\infty} f(t)e^{-j(-\omega)t}dt=F(-\omega)
\end{cases}
$$

8.若 $\mathbb{F}\left[f(t)\right]=F(\omega)$，证明（**三角性质**） $\begin{cases}\mathbb{F}\left[f(t)\cos\omega_0 t\right]=\dfrac{1}{2}\left[F(\omega+\omega_0)+F(\omega-\omega_0)\right] \\ \mathbb{F}\left[f(t)\sin\omega_0 t\right]=\dfrac{1}{2}j\left[F(\omega+\omega_0)-F(\omega-\omega_0)\right]\end{cases}$

证 明 ：

$$\begin{cases}\text{线性性质}\\ \text{及}\\ \text{象函数位移性质}\end{cases}\begin{cases}\mathbb{F}\left[f(t)\cos\omega_0 t\right]\\ =\mathbb{F}\left[f(t)\dfrac{e^{j\omega_0 t}+e^{-j\omega_0 t}}{2}\right]\\ =\dfrac{1}{2}\mathbb{F}\left[f(t)e^{j\omega_0 t}\right]+\dfrac{1}{2}\left[f(t)e^{-j\omega_0 t}\right]\\ =\dfrac{1}{2}\left[F(\omega-\omega_0)+F(\omega+\omega_0)\right]\end{cases}\begin{cases}\mathbb{F}\left[f(t)\sin\omega_0 t\right]\\ =\mathbb{F}\left[f(t)\dfrac{e^{j\omega_0 t}-e^{-j\omega_0 t}}{2j}\right]\\ =\dfrac{1}{2j}\mathbb{F}\left[f(t)e^{j\omega_0 t}\right]-\dfrac{1}{2j}\left[f(t)e\right.\\ =\dfrac{1}{2j}\left[F(\omega-\omega_0)-F(\omega+\omega_0)\right]\end{cases}$$

9. 若 $\mathbb{F}\left[f(t)\right]=F(\omega)$ a 为非零常数，证明（**原函数扩展相似性质**）

$$\mathbb{F}\left[f(at-t_0)\right]=\frac{1}{|a|}F(\frac{\omega}{a})e^{-j\frac{\omega}{a}t_0}$$

证明：

$$\text{定义证}\begin{cases}1.\text{当}a>0\text{时，}\\ \mathbb{F}\left[f(at-t_0)\right]=\int_{-\infty}^{+\infty}f(at-t_0)e^{-j\omega t}dt\\ (\text{令}at-t_0=u)=\int_{-\infty}^{+\infty}f(u)e^{-j\omega\frac{u+t_0}{a}}d\dfrac{u+t_0}{a}\\ \qquad=\dfrac{1}{a}e^{-j\frac{\omega}{a}t_0}\int_{-\infty}^{+\infty}f(u)e^{-j\frac{\omega}{a}u}du\\ (u\text{换成}t)\qquad=\dfrac{1}{a}e^{-j\frac{\omega}{a}t_0}\int_{-\infty}^{+\infty}f(t)e^{-j\frac{\omega}{a}t}dt\\ \qquad=\dfrac{1}{a}F(\dfrac{\omega}{a})e^{-j\frac{\omega}{a}t_0}\end{cases}\begin{cases}2.\text{当}a>0\text{时，}\\ \mathbb{F}\left[f(at-t_0)\right]=\int_{-\infty}^{+\infty}f(at-t_0)e^{-j\omega t}dt\\ (\text{令}at-t_0=u)=\int_{+\infty}^{-\infty}f(u)e^{-j\omega\frac{u+t_0}{a}}d\dfrac{u+t_0}{a}\\ \qquad=\dfrac{1}{a}e^{-j\frac{\omega}{a}t_0}\int_{+\infty}^{-\infty}f(u)e^{-j\frac{\omega}{a}u}du\Rightarrow\mathbb{F}\left[f(at)\right]=\dfrac{1}{|a|}F(\dfrac{\omega}{a})\\ (u\text{换成}t)=-\dfrac{1}{a}e^{-j\frac{\omega}{a}t_0}\int_{-\infty}^{+\infty}f(t)e^{-j\frac{\omega}{a}t}dt\\ \qquad=-\dfrac{1}{a}F(\dfrac{\omega}{a})e^{-j\frac{\omega}{a}t_0}\end{cases}$$

$$\begin{cases}\text{位移性质}\\ +\\ \text{相似性质}\end{cases}\begin{cases}\text{令}g(t)=f(at)\\ \mathbb{F}\left[f(at-t_0)\right]=\mathbb{F}\left[f\left(a(t-\dfrac{t_0}{a})\right)\right]=\mathbb{F}\left[g(t-\dfrac{t_0}{a})\right]=\mathbb{F}\left[g(t)\right]e^{-j\omega\frac{t_0}{a}}=\mathbb{F}\left[f(at)\right]e^{-j\frac{\omega}{a}t_0}=\dfrac{1}{|a|}F(\dfrac{\omega}{a})e^{-j\frac{\omega}{a}t_0}\end{cases}$$

卷积

定义：$f_1(t)*f_2(t)=\int_{-\infty}^{+\infty}f_1(\tau)f_2(t-\tau)d\tau$

不等性：$\left|f_1(t)*f_2(t)\right|\le\left|f_1(t)\right|*\left|f_2(t)\right|$

交换律：$f_1(t)*f_2(t)=f_2(t)*f_1(t)$
$$\begin{cases} f_1(t)*f_2(t)=\int_{-\infty}^{+\infty}f_1(\tau)f_2(t-\tau)d\tau \\ (令\ t-\tau=u)=-\int_{+\infty}^{-\infty}f_1(t-u)f_2(u)du \\ (令\ u=\tau)\qquad =\int_{-\infty}^{+\infty}f_2(\tau)f_1(t-\tau)d\tau=f_2(t)*f_1(t) \end{cases}$$

结合律：$f_1(t)*[f_2(t)*f_3(t)]=[f_1(t)*f_2(t)]*f_3(t)$
$$\begin{cases} f_1(t)*[f_2(t)*f_3(t)]=\int_{-\infty}^{+\infty}f_1(\tau)[f_2(t-\tau)*f_3(t-\tau)] \\ =\int_{-\infty}^{+\infty}f_1(\tau)[\int_{-\infty}^{+\infty}[f_3(u)\cdot f_2(t-\tau-u)du]d\tau \\ =\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}f_1(\tau)\cdot f_3(u)\cdot f_2(t-\tau-u)dud\tau \\ =\int_{-\infty}^{+\infty}\left[\int_{-\infty}^{+\infty}f_1(\tau)\cdot f_2(t-\tau-u)d\tau\right]f_3(u)du \\ =\int_{-\infty}^{+\infty}\left[\int_{-\infty}^{+\infty}f_1(t-u)*f_2(t-u)d\tau\right]f_3(u)du=[f_1 \end{cases}$$

分配律：
$$\begin{cases} f_1(t)*\left[f_2(t)+f_3(t)\right]=f_1(t)*f_2(t)+f_1(t)*f_3(t) \\ [f_1(t)+f_2(t)]*\left[g_1(t)+g_2(t)\right]=f_1(t)*g_1(t)+f_1(t)*g_2(t)+f_2(t)*g_1(t)+f_2(t)*g_2(t) \end{cases}$$

数乘律：$a[f_1(t)*f_2(t)]=[af_1(t)]*f_2(t)=f_1(t)*[af_2(t)]$

$$\frac{d}{dt}[f_1(t)*f_2(t)]=\frac{d}{dt}f_1(t)*f_2(t)=f_1(t)*\frac{d}{dt}f_2(t)=f_2(t)*\frac{d}{dt}f_1(t)$$

$$\begin{cases} \frac{d}{dt}[f_1(t)*f_2(t)]=\frac{d}{dt}\int_{-\infty}^{+\infty}f_1(\tau)f_2(t-\tau)d\tau=\int_{-\infty}^{+\infty}f_1(\tau)\frac{d}{dt}f_2(t-\tau)d\tau=f_1(t)*\frac{d}{dt}f_2(t) \\ \frac{d}{dt}[f_1(t)*f_2(t)]=\frac{d}{dt}[f_2(t)*f_1(t)]=\frac{d}{dt}\int_{-\infty}^{+\infty}f_2(\tau)f_1(t-\tau)d\tau=f_2(t)*\frac{d}{dt}f_1(t)=\frac{d}{dt}f_1(t)*f_2(t) \end{cases}$$

$f(t)*\delta(t)=f(t)$
$$\begin{cases} f(t)*\delta(t)=\int_{-\infty}^{+\infty}f(\tau)\delta(t-\tau)d\tau \\ \delta(t)为偶=\int_{-\infty}^{+\infty}f(\tau)\delta(\tau-t)d\tau \\ 筛选性质=f(\tau)\big|_{\tau=t}=f(t)变参 \end{cases}$$

$f(t)*\delta(t-t_0)=f(t-t_0)$
$$\begin{cases} f(t)*\delta(t-t_0)=\int_{-\infty}^{+\infty}f(\tau)\delta(t-t_0-\tau)d\tau \\ \delta(t)为偶=\int_{-\infty}^{+\infty}f(\tau)\delta\left(\tau-(t-t_0)\right)d\tau \\ 筛选性质=f(\tau)\big|_{\tau=t-t_0}=f(t-t_0)变参 \end{cases}$$

$f(t)*\delta'(t)=f'(t)$ $\left\{f(t)*\frac{d}{dt}\delta(t)=\frac{d}{dt}\left[f(t)*\delta(t)\right]=\frac{d}{dt}f(t)\right.$

$$f(t)*u(t)=\int_{-\infty}^{+\infty}f(\tau)u(t-\tau)d\tau$$

$$
\text{Laplase}\begin{cases}
\dfrac{\text{微分性质}}{\text{象函数}}\begin{cases}F^{(n)}(s)=\mathbb{L}^{(n)}\left[f(t)\right]=(-1)^{(n)}\mathbb{L}\left[t^n f(t)\right] \\ \begin{cases}\mathbb{L}\left[tf(t)\right]=-\mathbb{L}'\left[f(t)\right]=-F'(s) \\ f(t)=-\dfrac{1}{t}\mathbb{L}^{-1}\left[\mathbb{L}'\left[f(t)\right]\right]=-\dfrac{1}{t}\mathbb{L}^{-1}\left[F'(s)\right]\end{cases}\end{cases}
\begin{cases}F^{(n)}(s)=(-1)^n\mathbb{L}\left[t^n f(t)\right] \\ =\mathbb{L}\left[(-t)^n f(t)\right]=\int_0^{+\infty}(-t)^n f(t)e^{-st}dt \\ =\int_0^{+\infty}\dfrac{d^n}{ds^n}f(t)e^{-st}dt=\dfrac{d^n}{ds^n}\int_0^{+\infty}f(t)e^{-st}dt \\ =F^{(n)}(s) \qquad (\text{变换积分和微分顺序})\end{cases} \\
\dfrac{\text{积分性质}}{\text{象函数}}\begin{cases}\mathbb{L}\left[\dfrac{f(t)}{t}\right]=\int_s^{+\infty}F(s)ds \\ f(t)=t\mathbb{L}^{-1}\left[\int_s^{+\infty}F(s)ds\right]\end{cases}
\begin{cases}\int_s^{+\infty}F(s)ds=\int_s^{+\infty}\left[\int_0^{+\infty}f(t)e^{-st}dt\right]ds \\ \text{交换积分顺序}=\int_0^{+\infty}\left[\int_s^{+\infty}f(t)e^{-st}ds\right]dt \\ =\int_0^{+\infty}f(t)\left[-\dfrac{1}{t}e^{-st}\Big|_s^{\infty}\right]dt=\int_0^{+\infty}\left[\dfrac{f(t)}{t}e^{-st}\right]dt=\mathbb{L}\left[\dfrac{f(t)}{t}\right]\end{cases}
\end{cases}
$$

$$
\begin{cases}
(1)\, f(t)=te^{-3t}\sin 2t\to F(s)=? \\
(2)\, f(t)=t\int_0^t e^{-3t}\sin 2t dt\to F(s)=?\begin{cases}\mathbb{L}\left[f(t)\right]=-\mathbb{L}'\left[\int_0^t e^{-3t}\sin 2t dt\right] \\ =-\left(\dfrac{1}{s}\mathbb{L}\left[e^{-3t}\sin 2t\right]\right)'\end{cases} \\
(3)\, F(s)=\ln\dfrac{s+1}{s-1}\to f(t)=?\begin{cases}f(t)=-\dfrac{1}{t}\mathbb{L}^{-1}\left[F'(s)\right] \\ =-\dfrac{1}{t}\mathbb{L}^{-1}\left[\left(\ln\dfrac{s+1}{s-1}\right)'\right]=-\dfrac{1}{t}\mathbb{L}^{-1}\left[\dfrac{1}{s+1}-\dfrac{1}{s-1}\right] \\ =-\dfrac{1}{t}\left(e^{-t}-e^{t}\right)=\dfrac{2}{t}\cdot\dfrac{e^t-e^{-t}}{2}=\dfrac{2}{t}\sinh t\end{cases} \\
(4)\, f(t)=\int_0^t te^{-3t}\sin 2t dt\to F(s)=?\left\{\mathbb{L}\left[f(t)\right]=\dfrac{1}{s}\mathbb{L}\left[te^{-3t}\sin 2t\right]\right.
\end{cases}
$$

$$\begin{cases}(1)\,f(t)=\dfrac{\sin kt}{t}\to F(s)=?\begin{cases}\mathbb{L}\left[\dfrac{\sin kt}{t}\right]=\displaystyle\int_s^{\infty}\mathbb{L}\left[\sin kt\right]ds=\int_s^{\infty}\dfrac{k}{s^2+k^2}ds\\ =\arctan\dfrac{s}{k}\Big|_s^{\infty}=\dfrac{\pi}{2}-\arctan\dfrac{s}{k}=\operatorname{arc}\cot\dfrac{s}{k}\end{cases}\\ (2)\,f(t)=\dfrac{e^{-3t}\sin 2t}{t}\to F(s)=?\begin{cases}\mathbb{L}\left[\dfrac{e^{-3t}\sin 2t}{t}\right]=\displaystyle\int_s^{\infty}\mathbb{L}\left[e^{-3t}\sin 2t\right]ds=\int_s^{\infty}\dfrac{2}{(s+3)^2+4}ds\\ =\arctan\dfrac{s+3}{2}\Big|_s^{\infty}=\dfrac{\pi}{2}-\arctan\dfrac{s+3}{2}=\operatorname{arc}\cot\dfrac{s+3}{2}\end{cases}\\ (3)\,F(s)=\dfrac{s}{(s^2-1)^2}\to f(t)=?\begin{cases}f(t)=t\mathbb{L}^{-1}\left[\displaystyle\int_s^{+\infty}F(s)ds\right]\\ =t\mathbb{L}^{-1}\left[\displaystyle\int_s^{+\infty}\dfrac{s}{(s^2-1)^2}ds\right]=t\mathbb{L}^{-1}\left[\dfrac{1}{2}\bullet\dfrac{1}{s^2-1}\right]=\dfrac{t}{4}\left(e^{t}-e^{-t}\right)=\dfrac{t}{4}\sinh t\end{cases}\\ (4)\,f(t)=\displaystyle\int_0^t\dfrac{e^{-3t}\sin 2t}{t}dt\to F(s)=?\left\{\mathbb{L}\left[\displaystyle\int_0^t\dfrac{e^{-3t}\sin 2t}{t}dt\right]=\dfrac{1}{s}\mathbb{L}\left[\dfrac{e^{-3t}\sin 2t}{t}\right]=\dfrac{1}{s}\operatorname{arc}\cot\dfrac{s+3}{2}\right.\end{cases}$$