

Massachusetts Institute of Technology
Department of Electrical Engineering and Computer Science

6.685 Electric Machines

Class Notes 2 Magnetic Circuit Basics
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1 Introduction

Magnetic Circuits offer, as do electric circuits, a way of simplifying the analysis of magnetic field systems which can be represented as having a collection of discrete elements. In electric circuits the elements are sources, resistors and so forth which are represented as having discrete currents and voltages. These elements are connected together with ‘wires’ and their behavior is described by network constraints (Kirkhoff’s voltage and current laws) and by constitutive relationships such as Ohm’s Law. In magnetic circuits the lumped parameters are called ‘Reluctances’ (the inverse of ‘Reluctance’ is called ‘Permeance’). The analog to a ‘wire’ is referred to as a high permeance magnetic circuit element. Of course high permeability is the analog of high conductivity.

By organizing magnetic field systems into lumped parameter elements and using network constraints and constitutive relationships we can simplify the analysis of such systems.

2 Electric Circuits

First, let us review how Electric Circuits are defined. We start with two conservation laws: conservation of charge and Faraday’s Law. From these we can, with appropriate simplifying assumptions, derive the two fundamental circuit constraints embodied in Kirkhoff’s laws.

2.1 KCL

Conservation of charge could be written in integral form as:

$$\oint \vec{J} \cdot \vec{n} da + \int_{\text{volume}} \frac{d\rho_f}{dt} dv = 0 \quad (1)$$

This simply states that the sum of current *out* of some volume of space and rate of change of free charge in that space must be zero.

Now, if we define a discrete current to be the integral of current density crossing through a part of the surface:

$$i_k = - \iint_{\text{surface}_k} \vec{J} \cdot \vec{n} da \quad (2)$$

and if we assume that there is no accumulation of charge within the volume (in ordinary circuit theory the nodes are small and do not accumulate charge), we have:

$$\oint \vec{J} \cdot \vec{n} da = - \sum_k i_k = 0 \quad (3)$$

which holds if the sum over the index k includes all current paths into the node. This is, of course, KCL.

2.2 KVL

Faraday's Law is, in integral form:

$$\oint \vec{E} \cdot d\vec{\ell} = -\frac{d}{dt} \iint \vec{B} \cdot \vec{n} da \quad (4)$$

where the closed loop in the left hand side of the equation is the edge of the surface of the integral on the right hand side.

Now if we define *voltage* in the usual way, between points a and b for element k :

$$v_k = \int_{a_k}^{b_k} \vec{E} \cdot d\vec{\ell} \quad (5)$$

Then, if we assume that the right-hand side of Faraday's Law (that is, magnetic induction) is zero, the loop equation becomes:

$$\sum_k v_k = 0 \quad (6)$$

This works for circuit analysis because most circuits do not involve magnetic induction in the loops. However, it does form the basis for much head scratching over voltages encountered by 'ground loops'.

2.3 Constitutive Relationship: Ohm's Law

Many of the materials used in electric circuits carry current through a linear conduction mechanism. That is, the relationship between electric field and electric current density is

$$\vec{J} = \sigma \vec{E} \quad (7)$$

Suppose, to start, we can identify a piece of stuff which has constant area and which is carrying current over some finite length, as shown in Figure 1. Assume this rod is carrying current density \vec{J} (We won't say anything about how this current density managed to get into the rod, but assume that it is connected to something that can carry current (perhaps a wire....). Total current carried by the rod is simply

$$I = |\vec{J}|A$$

and then voltage across the element is:

$$v = \int \vec{E} \cdot d\vec{\ell} = \frac{\ell}{\sigma A} I$$

from which we conclude the *resistance* is

$$R = \frac{V}{I} = \frac{\ell}{\sigma A}$$

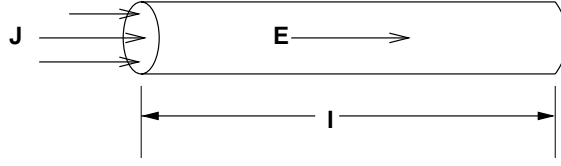


Figure 1: Simple Rod Shaped Resistor

Of course we can still employ the lumped parameter picture even with elements that are more complex. Consider the annular resistor shown in Figure 2. This is an end-on view of something which is uniform in cross-section and has depth D in the direction you can't see. Assume that the inner and outer elements are very good conductors, relative to the annular element in between. Assume further that this element has conductivity σ and inner and outer radii R_i and R_o , respectively.

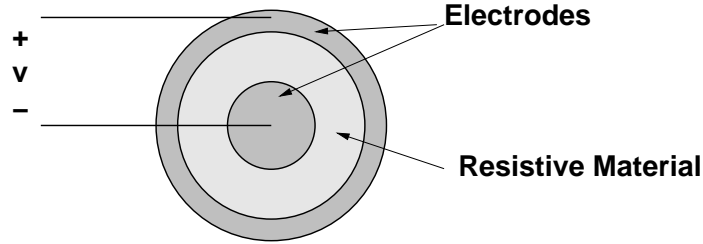


Figure 2: Annular Resistor

Now, if the thing is carrying current from the inner to the outer electrode, current density would be:

$$\vec{J} = \vec{i}_r J_r(r) = \frac{I}{2\pi D r}$$

Electric field is

$$E_r = \frac{J_r}{\sigma} = \frac{I}{2\pi D r \sigma}$$

Then voltage is

$$v = \int_{R_i}^{R_o} E_r(r) = \frac{I}{2\pi \sigma D} \log \frac{R_o}{R_i}$$

so that we conclude the resistance of this element is

$$R = \frac{\log \frac{R_o}{R_i}}{2\pi \sigma D}$$

3 Magnetic Circuit Analogs

In the electric circuit, elements for which voltage and current are defined are connected together by elements thought of as ‘wires’, or elements with zero or negligible voltage drop. The interconnection points are ‘nodes’. In magnetic circuits the analogous thing occurs: elements for which magnetomotive force and flux can be defined are connected together by high permeability magnetic circuit elements (usually iron) which are the analog of wires in electric circuits.

3.1 Analogy to KCL

Gauss’ Law is:

$$\oiint \vec{B} \cdot \vec{n} da = 0 \quad (8)$$

which means that the total amount of flux coming out of a region of space is *always* zero.

Now, we will define a quantity which is sometimes called simply ‘flux’ or a ‘flux tube’. This might be thought to be a collection of flux lines that can somehow be bundled together. Generally it is the flux that is identified with a magnetic circuit element. Mathematically it is:

$$\Phi_k = \iint \vec{B} \cdot \vec{n} da \quad (9)$$

In most cases, flux as defined above is carried in magnetic circuit elements which are made of high permeability material, analogous to the ‘wires’ of high conductivity material which carry current in electric circuits. It is possible to show that flux is largely contained in such high permeability materials.

If all of the flux tubes out of some region of space (‘node’) are considered in the sum, they must add to zero:

$$\sum_k \Phi_k = 0 \quad (10)$$

3.2 Analogy to KVL: MMF

Ampere’s Law is

$$\oint \vec{H} \cdot d\vec{\ell} = \iint \vec{J} \cdot \vec{n} da \quad (11)$$

Where, as for Faraday’s Law, the closed contour on the left is the periphery of the (open) surface on the right. Now we define what we call *Magnetomotive Force*, in direct analog to ‘Electromotive Force’, (voltage).

$$F_k = \int_{a_k}^{b_k} \vec{H} \cdot d\vec{\ell} \quad (12)$$

Further, define the current enclosed by a loop to be:

$$F_0 = \iint \vec{J} \cdot \vec{n} da \quad (13)$$

Then the analogy to KVL is:

$$\sum_k F_k = F_0$$

Note that the analog is not exact as there is a source term on the right hand side whereas KVL has no source term. Note also that *sign* counts here. The closed integral is taken in such direction so that the positive sense of the surface enclosed is positive (upwards) when the surface is to the left of the contour. (This is another way of stating the celebrated ‘right hand rule’: if you wrap your right hand around the contour with your fingers pointing in the direction of the closed contour integration, your thumb is pointing in the positive direction for the surface).

3.3 Analog to Ohm’s Law: Reluctance

Consider a ‘gap’ between two high permeability pieces as shown in Figure 3. If we assume that their permeability is high enough, we can assume that there is no magnetic field H in them and so the MMF or ‘magnetic potential’ is essentially constant, just like in a wire. For the moment, assume that the gap dimension g is ‘small’ and uniform over the gap area A . Now, assume that some flux Φ is flowing from one of these to the other. That flux is

$$\Phi = BA$$

where B is the flux density crossing the gap and A is the gap area. Note that we are ignoring ‘fringing’ fields in this simplified analysis. This neglect often requires correction in practice. Since the permeability of free space is μ_0 , (assuming the gap is indeed filled with ‘free space’), magnetic field intensity is

$$H = \frac{B}{\mu_0}$$

and gap MMF is just magnetic field intensity times gap dimension. This, of course, assumes that the gap is uniform and that so is the magnetic field intensity:

$$F = \frac{B}{\mu_0}g$$

Which means that the *reluctance* of the gap is the ratio of MMF to flux:

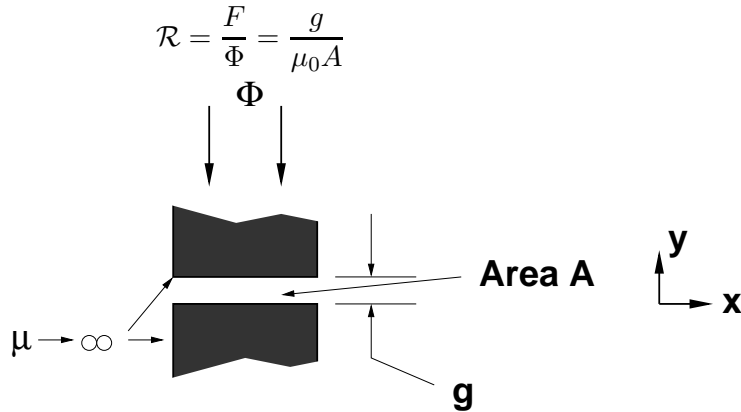


Figure 3: Air Gap

3.4 Simple Case

Consider the magnetic circuit situation shown in Figure 4. Here there is a piece of highly permeable material shaped to carry flux across a single air-gap. A coil is wound through the window in the magnetic material (this shape is usually referred to as a ‘core’). The equivalent circuit is shown in Figure 5.

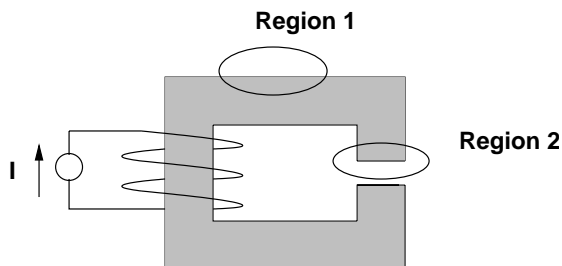


Figure 4: Single air-capped Core

Note that in Figure 4, if we take as the positive sense of the closed loop a direction which goes vertically upwards through the leg of the core through the coil and then downwards through the gap, the current crosses the surface surrounded by the contour in the positive sense direction.

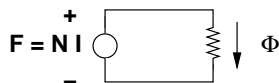


Figure 5: Equivalent Circuit

3.5 Flux Confinement

The gap in this case has the same reluctance as computed earlier, so that the flux in the gap is simply $\Phi = \frac{NI}{\mathcal{R}}$. Now, by focusing on the two regions indicated we might make a few observations about magnetic circuits. First, consider ‘region 1’ as shown in Figure 6.

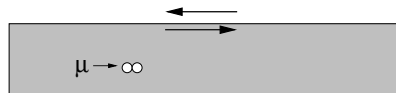


Figure 6: Flux Confinement Boundary: This is ‘Region 1’

In this picture, note that magnetic field \vec{H} parallel to the surface must be the same inside the material as it is outside. Consider Ampere's Law carried out about a very thin loop consisting of the two arrows drawn at the top boundary of the material in Figure 6 with very short vertical paths joining them. If there is no current singularity inside that loop, the integral around it must be zero which means the magnetic field just inside must be the same as the magnetic field outside. Since the material is very highly permeable and $\vec{B} = \mu\vec{H}$, and 'highly permeable' means μ is very large, unless B is *really* large, \vec{H} must be quite small. Thus the magnetic circuit has small magnetic field H and therefore flux densities parallel to and just outside its boundaries are also small.

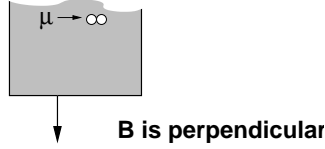


Figure 7: Gap Boundary

At the surface of the magnetic material, since the magnetic field parallel to the surface must be very small, any flux lines that emerge from the core element must be perpendicular to the surface as shown for the gap region in Figure 7. This is true for region 1 as well as for region 2, but note that the total MMF available to drive fields across the gap is the same as would produce field lines from the area of region 1. Since any lines emerging from the magnetic material in region 1 would have very long magnetic paths, they must be very weak. Thus the magnetic circuit material largely confines flux, with only the relatively high permeance (low reluctance) gaps carrying any substantive amount of flux.

3.6 Example: C-Core

Consider a 'gapped' c-core as shown in Figure 8. This is two pieces of highly permeable material shaped generally like 'C's. They have uniform depth in the direction you cannot see. We will call that dimension D . Of course the area $A = wD$, where w is the width at the gap. We assume the two gaps have the same area. Each of the gaps will have a reluctance

$$\mathcal{R} = \frac{g}{\mu_0 A}$$

Suppose we wind a coil with N turns on this core as shown in Figure 9. Then we put a current I in that coil. The magnetic circuit equivalent is shown in Figure 10. The two gaps are in series and, of course, in series with the MMF source. Since the two fluxes are the same and the MMF's add:

$$F_0 = NI = F_1 + F_2 = 2\mathcal{R}\Phi$$

and then

$$\Phi = \frac{NI}{2\mathcal{R}} = \frac{\mu_0 ANI}{2g}$$

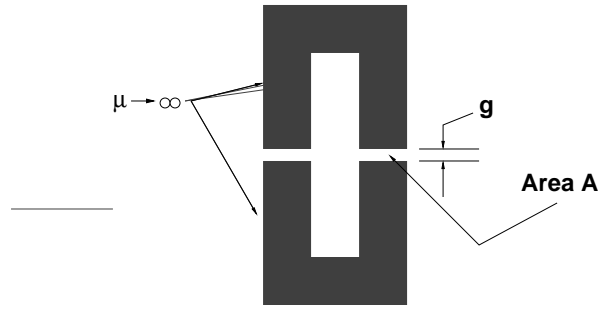


Figure 8: Gapped Core

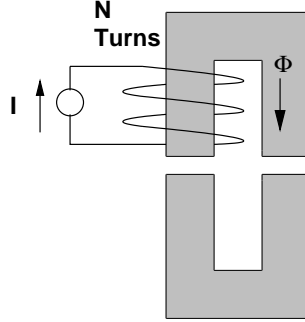


Figure 9: Wound, Gapped Core

and corresponding flux density in the gaps would be:

$$B_g = \frac{\mu_0 N I}{2g}$$

3.7 Example: Core with Different Gaps

As a second example, consider the perhaps oddly shaped core shown in Figure 11. Suppose the gap on the right has twice the area as the gap on the left. We would have two gap reluctances:

$$\mathcal{R}_1 = \frac{g}{\mu_0 A} \quad \mathcal{R}_2 = \frac{g}{2\mu_0 A}$$

Since the two gaps are in series the flux is the same and the total reluctance is

$$\mathcal{R} = \frac{3}{2} \frac{g}{\mu_0 A}$$

Flux in the magnetic circuit loop is

$$\Phi = \frac{F}{\mathcal{R}} = \frac{2}{3} \frac{\mu_0 A N I}{g}$$

and the flux density across, say, the left hand gap would be:

$$B_g = \frac{\Phi}{A} = \frac{2}{3} \frac{\mu_0 N I}{g}$$

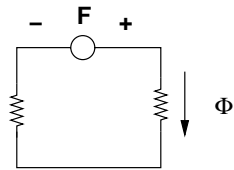


Figure 10: Equivalent Magnetic Circuit

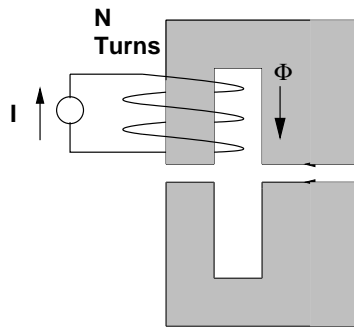


Figure 11: Wound, Gapped Core: Different Gaps